Summary and Abstracts of Bachelor's Thesis and Research Projects

Here is the abstract and summary of my bachelor's thesis, following by the abstract of my other four major research projects I have done during my undergraduate degree. The full pdf of them are available on my personal website (The documents are written in Persian) at: https://sherajafaritabar.wixsite.com/sherajafaritabar

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Bachelor’s Thesis: Black Hole’s Information Paradox (PDF)

Abstract:

In this thesis, I studied the black hole information paradox and reviewed several efforts made so far to solve it. Chapter one I began by taking a look at the history of black holes. In Chapter two I studied mathematical tools needed, and then looked at the similarities between the laws of classical thermodynamics and black holes. In the next chapter, I used Unruh effect to derive Hawking radiation, and the reason for this was to show the dependence of the concepts of particles and vacuum on the observer. Following that in chapter four, information and its relationship with entropy were introduced. In the last chapter, I studied the leading cause of the paradox, the information-free vacuum on the event horizon predicted by general relativity, which stretches as the black holes evaporates and produces Hawking pairs. The first solution we came up with was the remnant idea, which is that the black hole evaporates until it reaches the Planck size. We then studied to Page's theory, which states that after Page time, information begins to emerge from the black hole. Finally, I studied the idea of complementarity and an interesting article on entanglement and wormholes. The proposed solutions each had contradictions that was discussed: infinite degeneracy in remnant theory, loss of information in Page theory, and the existence of a firewall at the black hole’s even horizon in complementarity theory. The principle of equivalence of general relativity is in conflict.

Summary of chapter one:

Paradoxes that have a long history in the natural sciences and mathematics usually arise either from concepts that are not essentially paradoxical (such as the twin paradox of particular relativity) or from contradictions between theories. The black hole information paradox is the result of a contradiction between general relativity and quantum mechanics.

Stephen Hawking's paper suggests that information in a black hole will be lost forever once the black hole eventually evaporates completely. In quantum mechanics, the evolution of a quantum state must be determined by unitary operator. One of the attempts for preserving black hole’s evaporation was encoding the information in a Planck sized remnant. Another solution was, the idea of a complementarity that stated that information would both cross the event horizon and come back and get collected by an outside observer bit no observer can detect both. Nevertheless, in 2013 the AMPS "firewall" paradox showed that complementarity requires an energetic quantum layer on the event horizon; while the principle of equivalence of general relativity says that an observer does not feel anything when he falls on the horizon.

Summary of Chapter two:

Using two books of Carrol [1] and Wald [2], studied the mathematics behind black holes. Starting from Schwarzschild metric:

We found Riemann curvature Tensor for Schwarzschild black hole to be $K = \frac{48G^2M^2}{c^4r^6}$which diverges as $r \to 0$, which point out a singularity in the middle. [3] From equation 1 we see two singularities at $r = 0$ and $r = \frac{2GM}{c^2}$ which is Schwarzschild radius which is only a characteristic of metric. Schwarzschild is the unique solution to Einstein's field equations which is: [1]

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{2}$$

Then we moved on studying Rindler metric; which is within Minkowski to see what would metric look like for an eternally accelerating observer. Picking a new set of coordinated (equation 3) we found a region (region I) in Rindler space-time which is only accessible for an eternally accelerating observer (fig. 1) [4]

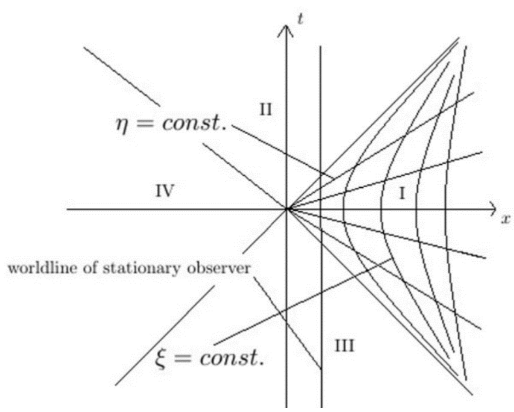


Figure 1

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2\,dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2\,d\theta^2 + \gamma^2\,sin^2\,\theta\,d\varphi^2 \tag{1}$$

$$t = \frac{1}{a}e^{a\xi}\sinh(a\eta)$$
$$x = \frac{1}{a}e^{a\xi}\cosh(a\eta) \tag{3}$$

Under new coordinates ξ and η, Minkowski metric will become Rindler metric:

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \tag{4}$$

The lines $x = \pm t$ are future and past null infinity, so they show the horizons and are different from black hole's event horizons.

For adding angular momentum and charge to the equation of Schwarzschild black holes, by changing Einstein-Maxwell action and finding the equation of motion, we will find: [4]

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 - \frac{2\,a\sin^2 \theta}{\Sigma}(r^2 + a^2 - \Delta) dt d\varphi + \Sigma\, d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} sin^2\,\theta\, d\varphi^2 \tag{5}$$

Where $\Sigma = r^2 + a^2 \cos^2 \theta$, and $\Delta = r^2 - 2Mr + Q^2 + a^2$

In case of $Q = 0$, we find Kerr metric which is not spherically symmetric, so Birkhoff's theorem will fail, so resulting in a not valid metric on the surface of a collapsing matter. After the space time turns to a stationary state, we have time translation symmetry or a timelike Killing vector. [4]

I went on studying mathematics used in black holes using Carrol [1]and Poisson [5]. By defining Lie derivative that acts on metric tensor, we found Killing vector as: [2]

$$(\mathcal{L}_\nu g)_{\mu\nu} = \nabla_\mu \nu_\nu + \nabla_\nu \nu_\mu = 0 \tag{6}$$

We used Killing horizons to define surface gravity as:

$$\chi^\mu \nabla_\mu \chi^\alpha = -\kappa \chi^\alpha \tag{7}$$

A stationary observer's 4-velocity: [1]

$$\kappa^\mu = V u^\mu \tag{8}$$

Where V is redshift factor and is defined as $\sqrt{-\kappa_\mu \kappa^\mu}$. Using charge conservation and calculating p_μ, we will get $\mathrm{Q} = -\mathrm{E} = -m\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$.

Knowing the frequency is given by $\omega = -p^\mu u_\mu$, we can write $E = -p^\mu k_\mu$. Using equation 8 we will have $\frac{E}{V} = \omega$. 4 acceleration $a^\nu = u^\alpha \nabla_\alpha u^\alpha$ can be defined as $a^\nu = u^\alpha \nabla_\alpha u^\alpha$ with the magnitude of $a = \frac{1}{V}\sqrt{\nabla_\nu V \nabla^\nu V}$ which equals infinity at the horizon with zero redshift factor. The surface gravity can be defined as [1]

$$\kappa = \mathrm{Va} \tag{9}$$

Calculating surface gravity in Schwarzschild space time will lead to the result [3]

$$\kappa = \frac{1}{4\mathrm{M}} \tag{10}$$

In the last section of chapter 2, I studied the Thermodynamics of black holes which in summary can be written as:

Law	Classical Thermodynamics	Black Holes
Zeroth	The temperature T is constant all through a system in thermal equilibrium	The surface gravity κ remains constant over the event horizon of a stationary black holes
First	$dE = TdS + W$	$dM = \frac{1}{8\pi}\kappa \mathrm{dA} + \Omega_{\mathrm{H}} dJ$
Second	The entropy S increases or stars the same in any process	The area A increases or stays the same in any process
Thrid	T = 0 can not be achieved in any physical process	κ = 0 cannot be achieved in any physical process

Summary of chapter 3

I started first with a short introduction to field theory using Lancaster Quantum Field Theory for the Gifted Amateur [6]
For a real massive Scalar field with Lagrangian $\mathcal{L} = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$. Deriving the equation of motion $\Box\phi = m^2\phi$, that gives us the plane wave solution:

$$\phi = \phi_0 e^{ip_u x^\mu} \tag{11}$$

We want to make a complete orthonormal basis for solutions consists of all positive frequency place wave model and their complex conjugates $\{\psi_p, \psi^* p\}$ [1]

$$\phi = \int d^3p\left(a_p\psi_P + a_p^\dagger\psi_P^*\right) \tag{12}$$

We define ψ_p as positive frequency ($\partial_t\psi_P = -i\omega\psi_P$) and ψ_p^* as negative frequency ($\partial_t\psi_p^* = i\omega\psi_p^*$). The coefficients are annihilation and creation operators satisfies: [1]

$$\begin{gathered} \left[a_p, a_{P'}\right] = 0 \\ \left[a_p^\dagger, a_{p'}^\dagger\right] = 0 \\ \left[a_p, a_{p'}^\dagger\right] = \delta^3(p - p') \end{gathered} \tag{13}$$

And in the vacuum state $a_P|0\rangle = 0$. Using the new inertial frame:

$$t = \gamma(t' + v.x') \tag{14}$$

$$x = \gamma(x' + vt')^1 \tag{15}$$

Using equation 11 and $\omega' = \gamma(\omega - v.k)$ we will find: [2]

$$\partial_{t'}\psi_p = \frac{\partial x^\mu}{\partial t'}\partial_\mu\psi_p = -i\omega'\psi_p \tag{16}$$

This shows that a quantum particle in a flat space, can be boosted to have a boosted momentum in the new frame. As a result, the notions of particle and vacuum are independent of observer in flat space. [7]

What about curved spaces? The same as the flat space we pick a complete set $\{f_i, f_i^*\}$ where: [4]

$$\phi = \sum_i \left(a_i f_i + a_i^\dagger f_i^*\right) \tag{17}$$

And the vacuum is defined as $a_i\left|0_f\right\rangle = 0$ and the excitation states can be found from $|n_i\rangle = \frac{1}{\sqrt{n_i!}}\left(a_i^\dagger\right)^{n_i}\left|0_f\right\rangle$. Though unlike flat space, $\{f_i, f_i^*\}$ is not unique. So we can do the same in another basis $\{g_i, g_i^*\}$ where $\phi = \sum_i \left(b_i g_i + b_i^\dagger g_i^*\right)$.

If we want to see how different observers observe a phenomenon, [1]

$$g_i = \sum_i \left(\alpha_{ij} f_i + \beta_{ij} f_i^*\right) \tag{18}$$

$$f_i = \sum_i \left(\alpha_{ij}^* g_j - \beta_{ij} g_j^*\right) \tag{19}$$

Where $\alpha_{ij} = (g_i, f_i)$ and $\beta_{ij} = -(g_i, f_j^*)$. Where: [1]

$$\begin{gathered}\sum_j \left(\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*\right) = \delta_{ij} \\ \sum_j \left(\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}\right) = 0\end{gathered} \tag{20}$$

Using Bogolubov transformation we find: [1]

$$a_i = \sum_i \left(\alpha_{ij} b_j + \beta_{jk}^* b_j^\dagger\right)$$

$$b_i = \sum_i \left(\alpha_{ij}^* \alpha_j - \beta_{ij}^* a_j^\dagger\right) \tag{21}$$

Now with quantum states in f-vacuum, we find:

$$\langle 0_f | n_{gi} | 0_f \rangle = \langle 0_f | b_i^\dagger b_i | 0_f \rangle = a_j^\dagger \left\langle 0_f \left| \sum\nolimits_{jk} \left(\alpha_{ij} a_j^\dagger - \beta_{ij} a_j\right)\left(\alpha_{ik}^* \alpha_k - \beta_{ik}^* a_k^\dagger\right) \right| 0_f \right\rangle$$
$$= \left(-\beta_{ij}\right)\left(-\beta_{ik}^*\right)\langle 0_f | a_j a_k^\dagger | 0_f \rangle = \beta_{ij}\beta_{ik}^* \delta_{jk} \langle 0_f | 0_f \rangle = \Sigma_j \beta_{ij} \beta_{ij}^* \neq 0 \tag{22}$$

This means that the notions of particle and vacuum are dependent on observers in curved space.

From adiabatic theory, we know that if potential changes slowly, will leave mod to stay in vacuum state. But if potential changes suddenly: [6]

$$|0\rangle_{\omega_1} = C|0\rangle_{\omega_2} + C_1|1\rangle_{\omega_2} + C_2|2\rangle_{\omega_2} + \cdots \tag{23}$$

Since the wave function is symmetric, $C_1 = C_3 = \cdots = 0$

$$|0\rangle_{\omega_1} = C|0\rangle_{\omega_2} + C_2|2\rangle_{\omega_2} + \cdots$$

As a result, a sudden change can result in an excited pair.

The next section is about the Unruh effect.

For an stationary observer in r > GM part of figure one, Rindler' s metric is:

$$ds^2 = e^{2a\xi}\left(-d\eta^2 + d\xi^2\right) \tag{24}$$

And there exist a killing vector: $\partial_\eta = \frac{\partial t}{\partial \eta}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} = a\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right)$ which is space like in II and III regions and timelike in I and IV. as a result $x = \pm t$ are killing horizons and in IV region we can have $x = -\frac{1}{a}e^{a\xi}\cos h(a\eta)$ and $t = -\frac{1}{a}e^{a\xi}\sin h(a\eta)$. We will have

$$\Box\phi = e^{-2a\xi}\left(\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \xi^2}\right)\phi = 0 \tag{25}$$

Can be solved by a plane wave mode $\omega = |k|$ and get

$$\frac{\partial}{\partial \eta} g_k = -i\omega g_k \tag{26}$$

The killing vector for positive frequencies must point to future and as a result, $\frac{\partial}{\partial\eta}$ is in opposite ways in I and IV. So we need two mods: [1]

In region I:

$$g_k^{(1)} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta+ik\xi} \tag{27}$$

In region IV:

$$g_k^{(2)} = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega\eta+ik\xi} \tag{28}$$

Now we will have:

$$\frac{\partial}{\partial\eta} g_k^{(1)} = -i\omega g_k^{(1)} \tag{29}$$
$$\frac{\partial}{\partial\eta} g_k^{(2)} = -i\omega g_k^{(2)}$$

Comparing to the same equations we had, we will see both are for the positive frequencies. These will make a complete set in space-time. We will have

$$\Phi = \int dk(b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*}) \tag{30}$$

Comparing 12 and 30 we can see that the vacuums all not the same. The vacuum observed by an observer in Minkowski space time $a_k|0_M\rangle = 0$, while for an observer in Rindler space time $b_k^{(1)}|0_R\rangle = b_k^{(2)}|0_R\rangle = 0$ which is full of particles and the Minkowski observer will not see vacuum.

Defining the coordinates as:

$$e^{-a(\eta-\xi)} = \begin{cases} a(-t+x) & I \\ a(t-x) & IV \end{cases} \tag{31}$$

$$e^{a(\eta-\xi)} = \begin{cases} a(t+x) & I \\ a(-t-x) & IV \end{cases} \tag{32}$$

Using 27 and 28, for $\omega = k$ we will find:

$$\sqrt{4\pi\omega} g_k^{(1)} = e^{-i\omega\eta+ik\xi} = e^{-i\omega(\eta-\xi)} = a^{\frac{i\omega}{a}}(-t+x)^{\frac{i\omega}{a}} \tag{33}$$

$$\sqrt{4\pi\omega} g_{-k}^{(2)*} = e^{i\omega\eta+ik\xi} = e^{i\omega(\eta-\xi)} = a^{\frac{i\omega}{a}}(t-x)^{\frac{i\omega}{a}} \tag{34}$$
$$= a^{\frac{i\omega}{a}}[e^{-i\pi}(-t+x)]^{\frac{i\omega}{a}} = a^{\frac{i\omega}{a}} e^{\frac{\pi\omega}{a}}(-t+x)^{\frac{i\omega}{a}}$$

We can have:

$$\sqrt{4\pi\omega}\left(g_k^{(1)} + e^{-\frac{\pi\omega}{2a}} g_{-k}^{(2)*}\right) = a^{\frac{i\omega}{a}}(-t+x)^{\frac{i\omega}{a}} \tag{35}$$

And taking we need to normalize 35, we will have

$$\phi = \int \mathrm{dk}\left(\mathrm{C}_\mathrm{k}^{(1)}\mathrm{h}_\mathrm{k}^{(1)} + \mathrm{C}_\mathrm{k}^{(1)^\dagger}\mathrm{h}_\mathrm{k}^{(1)^*} + \mathrm{C}_\mathrm{k}^{(2)}\mathrm{h}_\mathrm{k}^{(2)} + \mathrm{C}_\mathrm{k}^{(2)^\dagger}\mathrm{h}_\mathrm{k}^{(2)^*}\right) \tag{36}$$

And as we had before we can have:

$$b_k^{(1)} = \frac{1}{\sqrt{2\sin h\left(\frac{\pi\omega}{a}\right)}}\left(e^{\frac{\pi\omega}{2a}} c_k^{(1)} + e^{-\frac{\pi\omega}{2a}} c_{-k}^{(2)\dagger}\right) \tag{37}$$

$$b_k^{(2)} = \frac{1}{\sqrt{2\sin h\left(\frac{\pi\omega}{a}\right)}}\left(e^{\frac{\pi\omega}{2a}} c_k^{(2)} + e^{-\frac{\pi\omega}{2a}} c_{-k}^{(1)\dagger}\right)$$

In the end, for region I we will have:

$$\langle 0_M | n_R^1(k) | 0_M \rangle = \left\langle 0_M \middle| b_k^{(1)\dagger} b_k^{(1)} \middle| 0_M \right\rangle = \frac{1}{2\sin h\left(\frac{\pi\omega}{a}\right)}\left\langle 0_M \middle| e^{-\frac{\pi\omega}{2a}} c_{-k}^{(2)} c_{-k}^{(2)\dagger} \middle| 0_M \right\rangle$$

$$= \frac{e^{-\frac{\pi\omega}{2a}}}{2\sin h\left(\frac{\pi\omega}{a}\right)}\delta(0) = \frac{1}{e^{-\frac{2\pi\omega}{a}} - 1}\delta(0) \tag{38}$$

Which will have the expectation value of: [1]

$$T = \frac{a}{2\pi} \tag{39}$$

So an observer constantly accelerating in Minkowski vacuum experiences a thermal bath of particles.

The last part of this chapter is dedicated to evaporation of black holes. According to quantum mechanics, suggests that black holes eventually after a time that is bigger than universe's age will evaporate and leave a thermal radiation which will not contain any information. A very simple way of Hawking radiation is photon pairs which because of oscillations near the event horizon will appear. If this process happens near the event horizon, the one with energy –E will fall inside while the other will move to infinity where one observer will detect it as Hawking radiation. Taking two observers at r_1 and r_2 (infinity). For a static observer just outside the horizon, at $2M \gg \frac{1}{a_1}$ we see that the observer at r_1 experiences a flat space time, the observer at r_2 is not necessarily in a flar space time and will observe the Unruh radiation with redshift factor V.

$$T_2 = \frac{V_1}{V_2} T_1 \quad ; \quad V = \sqrt{\left(1 - \frac{2M}{r}\right)} \tag{40}$$

From 39 and 9, we will find

$$T_1 = \frac{\kappa}{2\pi} \tag{41}$$

And this is the temperature of Hawking radiation.

Summary of Chapter 4

At the beginning of this chapter, I started studying quantum information which was mostly done in more detail in my other research project prior to this. After that I moved on to the to studying what would happen if a bit falls into a black hole.

Using Planck-Einstein equation $E = hf$ and Einstein equation $E = mc^2$we find that $E = \frac{hc}{R_S}$. So the amount that the radius has increased equals $\Delta R_S = \frac{2Gh}{c^3 R_S}$, which gives $A = \frac{16\pi Gh}{c^3} = 16\pi \ell_p^2$, this will give us $S = \frac{A}{4}$ which is the famous Bekenstein's formula. [8]

Summary of Chapter 5

I started with describing the paradox using the method used in Mathur [9]. By introducing some nice spacelike slices that contains the black hole and goes inside the horizon so we can study Hawking pairs. These slices should be on an entirely spacelike slice where the intrinsic curvature $R^{(3)} \ll \frac{1}{l_P^2}$, have small extrinsic curvature $K \ll \frac{1}{l_P^2}$, in the neighborhood of the slice the four curvature must be small $R^{(4)} \ll \frac{1}{l_P^2}$, matter on the slice cannot approach Planck scale because of quantum gravity effects. (fig 2)

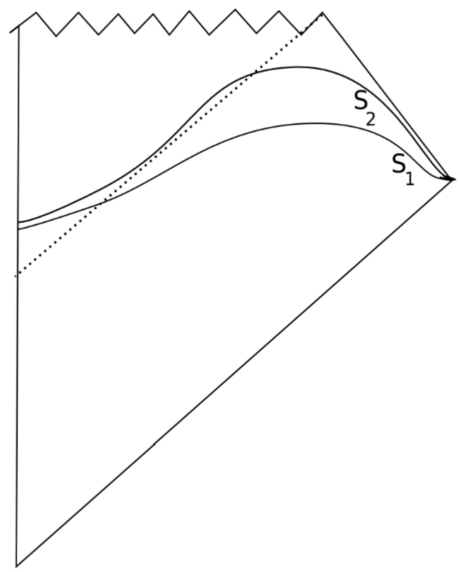


Figure 2: Penrose diagrams

As the slices change, the Hawking pairs will be created, taking b to be the particles that goes to infinity and will be detected as Hawking radiation, and c the particles that will fall into black holes, the entangled state can be written as: [10]

$$|\Psi\rangle \approx |\psi\rangle_{matter} \otimes |\psi\rangle_{pair} = \left(\frac{1}{\sqrt{2}}\Big|\uparrow\rangle_{matter} + \frac{1}{\sqrt{2}}\Big|\downarrow\rangle_{matter}\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_c|0\rangle_b + \frac{1}{\sqrt{2}}|1\rangle_c|1\rangle_b\right) \quad (42)$$

Dividing slices in three sections of inside, outside, and alongside the horizon, we see that a complete slice will be given by all of them, and as the parameters are shifting forward, they evolve one slice to another. According to fig 3, Σ_C will remain the same, while Σ_O shifts forward in time and Σ_I get longer. As they evolve needs to stretches to connect them together and this is what causes the Hawking pair. [10]

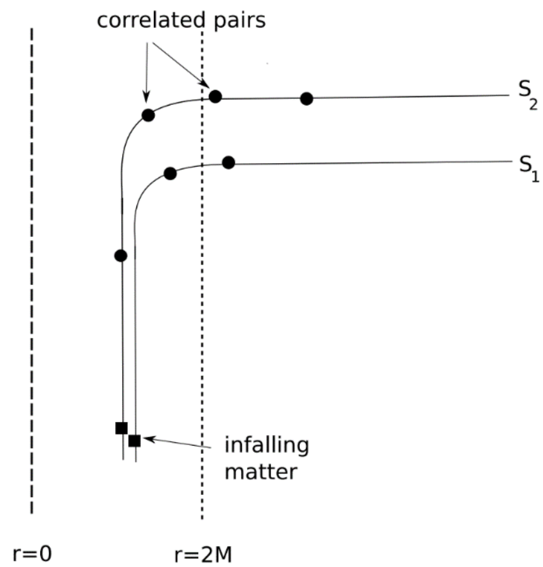


Figure 3: Evolution of Slices and the Black Hole

Now we want to see how the entanglement of Hawking's pair changes in time. We find that in the nth slice we will have

$$|\Psi\rangle = |\psi\rangle_{matter} \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_{c_1}|0\rangle_{b_1} + \frac{1}{\sqrt{2}}|1\rangle_{c_1}|1\rangle_{b_1}\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_{c_2}|0\rangle_{b_2} + \frac{1}{\sqrt{2}}|1\rangle_{c_2}|1\rangle_{b_2}\right) \otimes \dots \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_{c_n}|0\rangle_{b_n} + \frac{1}{\sqrt{2}}|1\rangle_{c_n}|1\rangle_{b_n}\right) \tag{43}$$

When black hole has radiated all its mass as Hawking radiation, it will leave only a radiation field that we have shown with b_1 ‘b_2 ‘ …‘b_n. Here the black hole has an entropy of $N \ln 2$ and was a mixed state. Before evaporation it was entangled with enteral states and had zero von was pure. As a result, this contradicts unitary. Then with few example I showed that a mixed state can contain all the information, and on the other hand, a final pure state can cause information loss.

After this part, I studied few given ideas and solutions. I started with an idea called "Remnant" which suggests that the evaporation will stop once it reaches the Planck's length. But since the remnant is entangled with entropy of $N \ln 2$ to the radiation, we need minimum N internal state with finite energy, however arbitrary N can cause arbitrary degeneracy. [10], [11]

The next theory is "Bleaching" which suggests that the horizons bleaches the information from getting inside. This will contradict the emptiness of horizon which was assumed before. [12]

The next theory, was Page theory. I started with finding the average Information for a system as: [13], [14]

$$I_{m,n} \equiv S_{max} - \langle S_A \rangle = \ln m - S_{m,n} \tag{44}$$

And using Mathematics page found the equation below (Although he could not prove it. However, there is a written proof in my thesis for this equation.)

$$S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} \tag{45}$$

For the $1 \ll m \leq n$ Page will approximate $S_{m,n}$ as

$$S_{m,n} = lnm - \frac{m}{2n} \tag{46}$$

If B is the black hole system, and R radiation and BR is the final pure system, we will find:

$$\begin{aligned} \langle I_R \rangle = \ln m - S_R &= \ln m - \ln n + \left(\ln n + \frac{n-1}{2m} - \sum_{k=n+1}^{mn} \frac{1}{k} \right) \\ &= \ln m + \frac{n-1}{2m} - \sum_{k=n+1}^{mn} \frac{1}{k} \approx \ln m - \ln n + \frac{n}{2m} \end{aligned} \tag{47}$$

$$\langle S_R \rangle = \ln n - \left(\ln n + \frac{n-1}{2m} - \sum_{k=n+1}^{mn} \frac{1}{k} \right) = \frac{n-1}{2m} - \sum_{k=n+1}^{mn} \frac{1}{k} \approx \ln n + \frac{n}{2m}$$

And given mn=291500 we will have: [13], [14]

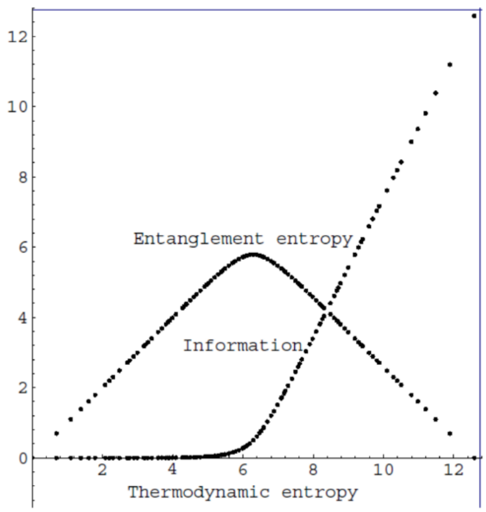


Figure 5: Average von Neumann Entropy versus Entanglement Entropy

We see that when has a smaller Hilbert spatial dimension remains than a black hole, this radiation usually contains very little information and is maximally mixed. Now consider the case where the black hole radiates most of its energy so that the radiation has larger dimensions. If one examines only part of the radiation, he will see only a very small amount of information in separate parts. Information instead is in the interaction between all the parts. We can also see that information begins to emerge after about half the entropy of a black hole has evaporated. Half of the initial entropy is the time it takes for a black hole to reach from the initial state to the point where it begins to radiate information, which is called "page time".

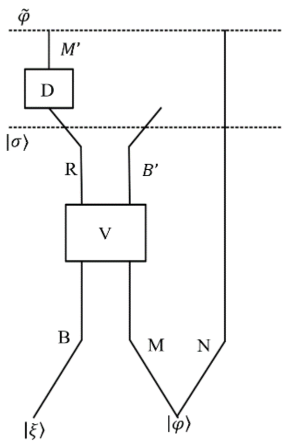
This time point is clearly visible in Figure 5.3. A black hole that has passed its page time is called an old black hole. As a result, no information is available until the fixed part of the black hole entropy has been destroyed, which means that [13], [14]

$$t_{info} \sim Black\ hole's\ lifetime \sim R^3 \tag{48}$$

This part is inspired from Patrick Hayden's lectures. [15], [16], [17]. Next, an example on Bob and Alice, and what would happen if Alice falls into black hole and Bob collects Hawking radiation. Then Bob also goes into black hole and Alice wants to compare his results with Bob.

$$\tau \sim R \exp(\frac{-\Delta t}{R}) \xrightarrow{\tau > \frac{1}{M}} \Delta t < R \log\ (MR) \xrightarrow{M \sim R} \Delta t < R \log\ (R) \tag{49}$$

So we need $t_{info} > R\, log\, R$ so that Alice and Bob will not compare their results and no-cloning theorem will not be violated. But on the other hand we still have Page's theorem that information will not come out. Following this, we will study an important example which is



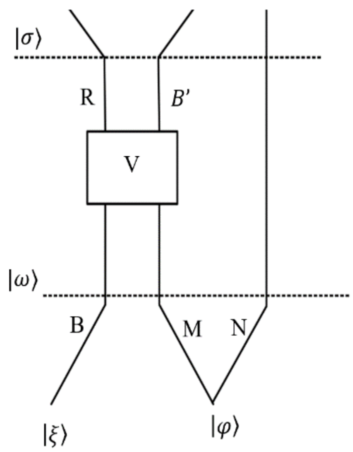
we find that in order for D to exist so we can decode, we need:

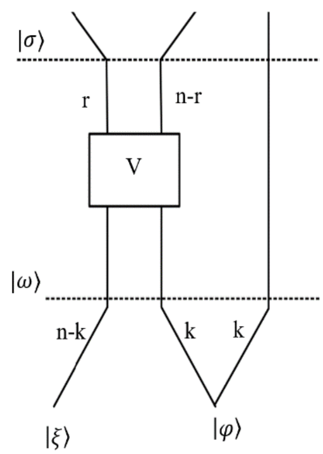
$$\sigma_{B'N} = \sigma_{B'} \otimes \Phi_N \tag{50}$$

Now we want to try to find a mutual information between N and M

B is a pure system, this means that eigenvalues of BM equals eigenvalues of M and mutual information is a function of entropy and entropy is a function of Eigen functions so we can add B to M:

$$I(N;M)_\omega = I(N;BM)_\omega = I(N;RB')_\sigma \tag{51}$$

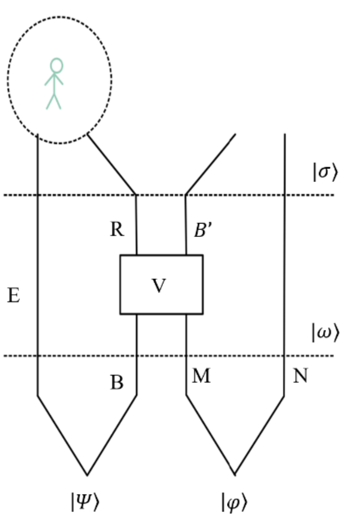




after using equation 50 and Renyi entropy we will find:

$$\int \| \sigma_{B'N} - \mathcal{L}_{\mathrm{B'}} \otimes \phi_{\mathrm{N}} \|^2 \, dV \leq 2^{S_0(N)_\omega} \times 2^{S_0(BM)_\omega} \times 2^{-S_2(BMN)} \times 2^{-2r} \sim 2^{I(N;BM)-2r} \tag{52}$$

This means that the correlation between N and BM wil decrease when information falls into black hole and if we have enough information we will get zero correlation. This means that $\mathrm{S}_0(\mathrm{N})_\omega$ is maximally mixed and $S_0(N)_\omega = \log rank\ [|\xi_0\rangle\langle\xi_0| \otimes \phi_{\mathrm{N}}] = k$. Now we need to maximize the rank. For this purpose, we will use an old black hole which is already entangled with its own radiation. And again equation 50 must hold for us to be able to decode. For an old black hole we will have:

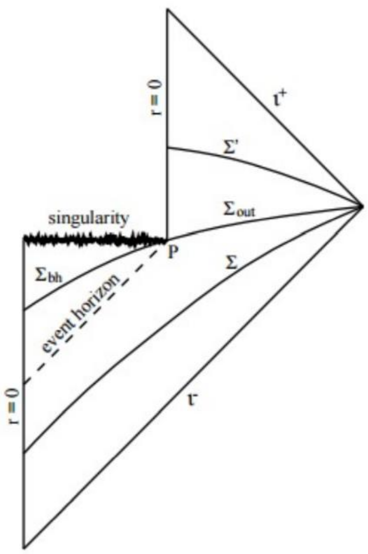


$$\int \| \sigma_{B'N} - \mathcal{L}_{\mathrm{B'}} \otimes \phi_{\mathrm{N}} \|^2 \, dV \leq \frac{d_N d_{BM}}{d_R^2}\ tr\,[(\omega_{BMN})^2] = \frac{2^k 2^n}{2^{2r}} 2^{-(n-k)} = 2^{2(k-r)} \tag{53}$$

So we would be able to decode if $r < k$. From equation 49 we will have only when $t_{info} < r_s \log r_s$ Bob can go inside the black hole and compare his result, and this will contradicts complementarity theorem.

The last part of this research was complementarity theorem. This theory states that when an object falls inside a black hole, it will hit stretched horizon which is one Planck length above Schwarzschild hoizon, which will radiate it as Hawking radiation. So this object will be seen inside from the point of view of an observer falling inside the black hole and will be seen outside from the observer outside, but no one sees both at the same time.

The slices will all satisfy the conditions mentioned previously. We define a Cauchy surface which $\Sigma_P = \Sigma_{bh} \cup \Sigma_{out}$. According to the last condition, $|\psi(\Sigma)\rangle$ can smoothly change to $|\psi(\Sigma_p)\rangle$ and then $|\psi(\Sigma')\rangle$ (Σ' is the space after evaporation of black hole. If a unitary matrix S acts on $|\psi(\Sigma)\rangle$ gives $|\psi(\Sigma')\rangle$, and $|\psi(\Sigma')\rangle$ needs to be a pure state. Also $|\psi(\Sigma')\rangle$ is the result of the evolution of $|\xi(\Sigma_{out})\rangle$. This means that $|\psi(\Sigma_p)\rangle = |\Pi(\Sigma_{bh})\rangle \otimes |\xi(\Sigma_{out})\rangle$. [18]



This product is the result of evolution of $|\psi(\Sigma)\rangle$ and so is $|\xi(\Sigma_{out})\rangle$. So it seems like $|\Pi(\Sigma_{bh})\rangle$ is independent of $|\psi(\Sigma)\rangle$ or in other words, it is bleaching the information. Complementarity contradicts this by stating that Σ_p cannot simultaneously be at both inside and outside of the black hole. [18]

Since Hawking pair only is created in vacuum, and not in a stretched horizon, there is a non-local physics in Schwarzschild radius. According to AMPS, we need a firewall in horizon, which contradicts equivalence principle. This leads to a non-physical firewall. [12]

An observer from distance receives Hawking radiation, while another observer falling into a black hole experiences. For the first observer, Hawking's initial radiation is farther away from the black hole, while the newly emitted radiation is in the stretched horizon. Second observer measures the vacuum on the horizon. On the other hand, the entropy entanglement between the black hole and the radiation for observer A increases steadily, which contradicts Page's theory, since he only can measure the entropy entanglement after Half of the black hole is evaporated. [19]

ER = EPR. ER = EPR, (Einstein-Podolski-Rosen paradox) and wormholes (also known as Einstein-Rosen bridges) states that Hawking pairs are related through wormholes and therefore are not independent systems, allowing them to interact with each other. ER = EPR does not rule out the existence of firewalls, but states that if it can be shown that the Einstein-Rosen Bridge connecting the black hole to its radiation is smooth near the black hole, then there would be no firewall. [20]

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Research Project: Thermodynamics of a Black Hole (PDF)

Abstract:

In this project, I worked on mathematical derivation on important results of classical black hole thermodynamics, for example the area increase theorem in the second law, Bardeen-Carter-Hawking formulation in second law, zeroth law and third law in the end. We have seen that three laws of thermodynamics are similar to four laws of thermodynamics:

- Zeroth law states that surface gravity is constant throughout the event horizon of a stationary black hole. (Surface gravity similar to temperature)
- First law is an energy conservation statement which gives a formula for the change in black hole's mass.
- For the second law we discussed how Bekenstein found his famous formula that a black hole's entropy is proportional to the event horizon area and the generalized second law of thermodynamics which is the sum of black hole's entropy and entropy in universe will never decrease
- In third law, states that surface gravity cannot be reduced to zero by any infinite sequence of operations similar to Nernst equation.

After that I studied how classical black hole thermodynamics is not complete. The reason for this is that black holes that are described completely by general relativity, do not radiate and their temperature is absolute zero. Following that, using semi classical WKB technique we saw that black holes can radiate (this result is also discussed in my other research project (*Hawking Radiation and Quantum Tunneling*)). This happens because of the non-zero thermodynamic temperature that they have which is proportional to their surface gravity, which means that the radiation is described through tunneling effect and is not completely thermal as Hawking previously had thought.

Research Project: Hawking Radiation as Quantum Tunneling (PDF)

Abstract:

In this project, I started from Einstein field equation and Schwarzschild solution. Then introduced Eddington-Finkelstein coordinates which was a better coordinate for studies near and through the event horizon. Using the WKB method, we explained how particles tunnel and we found a rate of tunneling which in massless self-gravitating shells gives:

$$\Gamma \propto e^{-\frac{8\pi EM}{\hbar}\left(1-\frac{E}{2M}\right)}$$

For small E we found the temperature to be:

$$T = \frac{\hbar c^3}{8\pi k_B GM}$$

Which matches with what Hawking had predicted. So it seems that tunneling through horizon is a reasonable process.

One other process I researched about was information loss during this process. Since all the particles can enter the black hole, we have a finite amount of energy coming inside the system. But if the black hole radiates thermally, a large portion of the information will be destroyed. But this is true if and only if it is thermal radiation. If its thermal we should have a Boltzmann factor of $e^{-\beta E}$. But in our calculations we found that we have an extra term of $1 - \frac{E}{2M}$. This is enough to tell us that the radiation is not purely thermal, thus it contains information about the main particle. This shows that the black hole can radiate and lose no information.

Research Project: Information and Entropy (PDF)

Abstract:

In this relatively short project, I researched on the nature of black hole entropy. Entropy can be a part of its thermodynamics since we have seen that black holes' thermodynamic temperature is not zero. I studied that how black holes have negative heat capacity which means that when matters fall inside a black hole, it will get colder, and as result it will not get to a thermal equilibrium. This can hinder our definition of thermodynamic entropy; because if we want to relate entropy of a black hole to thermodynamics, we need to have a thermal equilibrium.

To solve this problem, we needed a new definition for entropy which could also be true for non-equilibrium, which was von Neumann entropy which has a classical analog known as Shannon entropy. I also studied that how entropy is basically measuring how much we do not know about a system.

Following this, I examined what would happen to a harmonic oscillator falling into a black hole and noted that second law holds and the increase of black hole entropy is bigger than Shannon entropy of the oscillator. This result showed that inserting Shannon (von Neumann) Entropy in second law, for a generalized second law is a reasonable decision.

After this part, I briefly studied some of the theories that have been made for identifying the degrees of freedom responsible for black hole entropy, Information paradox, and entropy bounds. Finally, I focused I briefly looked at two ideas; entropy and black hole stability, and thermodynamic description of interaction between the black hole and matter.

Research Project: Understanding Spin through Symmetries (PDF)

Abstract:

In this project, I studied how symmetries are important in Physics. Starting from showing that action S for a point particle under Poincare transformation is invariant. Then I derived the relativistic action, using relativistic distance ds, between t_a and t_b. By applying nonrelativistic limit and setting it equal to the nonrelativistic action we found that:

$$S = -m \int_{t_a}^{t_b} \sqrt{\dot{x}^2} dt \tag{1}$$

I also showed that the same also applies for the equations of motion; the nonrelativistic limit of the equations of motion for relativistic action, must be equal to nonrelativistic equations of motion.

Noether's theorem states that there is a conserved quantity for every symmetry that exists. I studied Noether's theorem in field theory and found that Noether's current, j^μ, and Noether charge, Q, are invariant if and only if the Lagrangian density, $\mathcal{L}$, is symmetric under a transformation in 4-vector x^μ and field ϕ

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \eta_k - T^{\mu\nu} \xi_\nu$$
$$Q = \int d^3\bar{x} j^0 \tag{2}$$

$T^{\mu\nu}$ is energy-momentum tensor, η_k are infinitesimal generators for transformation in the field ϕ, ξ^μ infinitesimal generator for the four-vector x^μ. Using equation 2, we got to a modified Noether current, $\mathcal{J}^{\mu,\nu\rho}$, which is known as total angular momentum density, and the total angular momentum $j^{\nu\rho}$ Noether charge for $\mathcal{J}^{\mu,\nu\rho}$, which means that it will be invariant under Lorentz transformation. However, we showed that it is not invariant under translation, and so it is not invariant under all Poincare transformation, so we introduced Pauli-Lubanski spin vector:

$$W_\mu \coloneqq \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{3}$$

I showed that Pauli-Lubanski spin is invariant under Poincare group transformation, moreover in a particle's rest of frame, it is proportional to the total angular momentum.

Then Casmir operator, W^2, was introduced. If Ψ represents physical states and s, spin quantum number we will have:

$$W^2 \Psi = m^2 s(s+1) \Psi \tag{4}$$

Since W^2 is as spin of a particle in quantum field theory, and in a particle's rest of frame is proportional to the total angular momentum, equation 3 will be as a spin of a particle in classical field theory.

Next using reparametrization I fixed an invariance gauge for classical bosonic string. After one string dimension and embedding dimension were compactified, we got the action for a bosonic rigid particle (m is the mass, α is the rigidity)

$$S = -\frac{1}{2}\int d\tau \sqrt{-\dot{x}^2}\left(m + \frac{1}{-\alpha\dot{x}^2}[\dot{x}^2 y^2 + 2\dot{x}\dot{y} - (\dot{x}\dot{y})^2]\right) \tag{5}$$

equation 5 describes a relativistic bosonic rigid particle because can be written as:

$$S = \int d\tau \sqrt{-\dot{x}^2}\,(m + \alpha^{-1}k^2) \tag{6}$$

Where k is the extrinsic curvature. We saw that equation 5 is an action for a smooth spinning particle with only integer spin. For half-integer one should study supersymmetry.